Существование классического решения краевой задачи

(1) $u_{tt} = a^2 u_{xx}, \quad 0 < x < l, \quad t > 0$

(2) $u(x,0) = \varphi(x), \quad x \in [0,l]$
$u_t(x,0) = \psi(x)$

(3) $u(0,t) = u(l,t) = 0 \quad t \geq 0$

(8) $$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{a\pi n t}{l} + B_n \sin \frac{a\pi n t}{l} \right) \sin \frac{n\pi x}{l}$$

(10) $$A_n = \frac{2}{l} \int_0^l \varphi(x) \sin \frac{n\pi x}{l} dx, \qquad B_n = \frac{2}{a\pi n} \int_0^l \psi(x) \sin \frac{n\pi x}{l} dx \quad (11)$$

$n = 1, 2, \ldots$

Теорема. $\exists\ \varphi(x), \psi(x)$ (1)-(3):

1) $\varphi(x) \in C^2([0,l])$, $\varphi'''(x)$ - кус.-непр.

2) $\psi(x) \in C^1([0,l])$, $\psi''(x)$ - кус.-непр.

3) $\varphi(0) = \varphi(l) = \varphi''(0) = \varphi''(l) = 0$

$\psi(0) = \psi(l) = 0$

$\exists$ (1)-(3), (8), (10), (11)

1. Обобщен. принцип суперпозиции

$\exists\ Lu\ ,\quad u_i\ \ (i=1,2,\dots)\qquad Lu=0^{(*)},\qquad u=\sum_{i=1}^{\infty} c_i u_i$

2. $\sum_{i=1}^{n} c_i L u_i$

3. Признак Вейерштрасса. $\underbrace{\sum_{i=1}^{\infty} u_i}_{\text{непр}} \to \sum_{i=1}^{\infty} \tilde{u}_i$

4. равн. сх, $\to$ непр.

5. с $2l$ $\quad F(x) \quad \underline{\underline{k}},\ k+1 \qquad \sum_{n=1}^{\infty} n^k\left(|a_n|+|b_n|\right)$, сх.

$\sin\frac{k\pi x}{l} \qquad f(x) \qquad [0,l] \qquad F(x)$

$F(x) \qquad f(0)=0\ ,\ f(l)=0$

$\underline{f^{(k)}(0)=f^{(k)}(l)=0,\quad k=0,2,4,\dots}$

Док-во:

1. $$u(x,t)=\sum_{n=1}^{\infty}\left(A_n\cos\frac{an\pi t}{l}+B_n\sin\frac{an\pi t}{l}\right)\sin\frac{n\pi x}{l} \qquad 2.$$

$$u_n(x,t)=A_n\cos\frac{an\pi t}{l}+B_n\sin\frac{an\pi t}{l}$$

$$|u_n|\le|A_n|+|B_n|$$

$$\sum_{n=1}^{\infty}\left(|A_n|+|B_n|\right)\ \text{— сх.}$$

$$A_n=\frac{2}{l}\int_0^l\varphi(x)\sin\frac{n\pi x}{l}\,dx,\qquad B_n=\frac{2}{a\pi n}\int_0^l\psi(x)\sin\frac{n\pi x}{l}\,dx$$

$$\underbrace{\sum_{n=1}^{\infty}|A_n|}_{\text{сх.}}$$

$$b_n=B_n\cdot\frac{a\pi n}{l}=\frac{2}{l}\int_0^l\psi(x)\sin\frac{n\pi x}{l}\,dx$$

$$|b_n|\qquad\sum_{n=1}^{\infty}|b_n|$$

$$\underbrace{\sum_{n=1}^{\infty}|B_n|\frac{a\pi n}{l}}_{\text{сх.}}\ngeq\underbrace{\sum_{n=1}^{\infty}|B_n|\frac{a\pi}{l}}_{\text{сх.}}$$

$$\boxed{B_n=\frac{b_n\, l}{a\pi n}}$$

2. u_t - непр ?

$$u_t = \sum_{n=1}^{\infty}\left(-A_n\frac{an\pi}{l}\sin\frac{an\pi t}{l} + B_n\frac{an\pi}{l}\cos\frac{an\pi t}{l}\right)\sin\frac{n\pi x}{l}$$

$$\sum_{n=1}^{\infty}\left(|A_n|\frac{an\pi}{l} + |B_n|\frac{an\pi}{l}\right) = \frac{a\pi}{l}\left(\sum_{n=1}^{\infty}\left(n|A_n| + n|B_n|\right)\right)$$

$\sum_{n=1}^{\infty} n|A_n|$ - сх. $\qquad \sum_{n=1}^{\infty}|B_n|n = \frac{l}{a\pi}\sum_{n=1}^{\infty}|\beta_n|$

3. $$u_x = \sum_{n=1}^{\infty}\left(A_n\cos\frac{an\pi t}{l} + B_n\sin\frac{an\pi t}{l}\right)\cos\frac{n\pi x}{l}\cdot\frac{n\pi}{l}$$

$$\frac{\pi}{l}\sum_{n=1}^{\infty}\left(n|A_n| + n|B_n|\right) \text{ - сх.}$$

4. $$u_{xx} = -\sum_{n=1}^{\infty}\left(\frac{n\pi}{l}\right)^2\left(A_n\cos\frac{a\pi n t}{l} + B_n\sin\frac{an\pi t}{l}\right)\sin\frac{n\pi x}{l}$$

$$u_{tt} = -\sum_{n=1}^{\infty}\left(\frac{an\pi}{l}\right)^2\left(A_n\cos\frac{a\pi n t}{l} + B_n\sin\frac{an\pi t}{l}\right)\sin\frac{n\pi x}{l}$$

$\rightarrow \dfrac{\text{непр.}}{u_{xx},\ u_{tt}}$

$$\vee\ \cancel{\forall}\ \sum_{n=1}^{\infty} n^2\left(A_n\cos\frac{a\pi n t}{l} + B_n\sin\frac{an\pi t}{l}\right)\sin\frac{n\pi x}{l}$$

равн.-сх.

$$\sum_{n=1}^{\infty} n^2\left(|A_n| + |B_n|\right) \text{ — сх.}$$

$$\sum_{n=1}^{\infty} n^2|B_n| = \sum_{n=1}^{\infty} n^2\left|\frac{l}{\pi a n}\, b_n\right| = \left(\frac{l}{\pi a}\right)\underbrace{\sum_{n=1}^{\infty} n\,|b_n|}_{\text{сх.}}$$

$$\underbrace{\sum_{n=1}^{\infty} n^2|A_n|}_{\text{сх.}}\quad (k=2)$$

$b_n =$

$\sum$

$\varphi(x)$, A_n

5. $$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

u_n: (1) , $n = 1, 2, \ldots$