$$
\begin{cases}
(1)\quad u_{tt} = a^2 u_{xx}, \quad 0 < x < \ell,\ t > 0 \\
(2)\quad u(x,0) = \varphi(x),\quad u_t(x,0) = \psi(x), \quad 0 \le x \le \ell \\
(3)\quad u(0,t) = u(\ell,t) = 0, \quad t \geq 0
\end{cases}
$$

$$
(8)\quad u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{a\pi n t}{\ell} + B_n \sin \frac{a\pi n t}{\ell} \right) \sin \frac{n\pi x}{\ell}
$$

$$
A_n = \frac{2}{\ell} \int_0^{\ell} \varphi(x) \sin \frac{n\pi x}{\ell}\, dx, \qquad n = 1, 2, \ldots
$$

$$
B_n = \frac{2}{a\pi n} \int_0^{\ell} \psi(x) \sin \frac{n\pi x}{\ell}\, dx,
$$

$$
(*)\quad Lu = 0, \quad u_1, u_2 \ldots \qquad L = \frac{\partial^2}{\partial t^2} - a^2 \frac{\partial^2}{\partial x^2}
$$

$$
\sum_{n=1}^{\infty} u_n
$$

$$
u(x,0) \overset{(8)}{=} \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{\ell} = \varphi(x)
$$

$$
u_t(x,0) \overset{(8)}{=} \sum_{n=1}^{\infty} B_n \cdot \frac{a\pi n}{\ell} \cdot \sin \frac{n\pi x}{\ell} = \psi(x)
$$

$T'' \cdot X = a^2 X'' \cdot T$

T'

$$\begin{cases} u_{tt} = a^2 u_{xx} \\ u(0,t) = u(l,t) = 0 \end{cases}$$

$u = X(x)T(t) \not\equiv 0$

$\frac{T''}{a^2 T} = \frac{X''(x)}{X} = -C$

$X''(x) + C X(x) = 0$

$X(0) \cdot T(t) = X(l) \cdot T(t) = 0$

Вынужденные колебания струны с жестко закрепленными концами

(1) $u_{tt} = a^2 u_{xx} + f(x,t)$, $0 < x < l$, $t > 0$

(2) $u(x,0) = u_t(x,0) = 0$, $0 \le x \le l$,

(3) $u(0,t) = u(l,t) = 0$, $t \ge 0$

(4) $u(x,t) = \sum_{n=1}^{\infty} T_n(t) X_n(x) = \sum_{n=1}^{\infty} T_n(t) \cdot \sin\frac{n\pi x}{l} \rightarrow (1)$

$$\begin{cases} X''(x) + C X(x) = 0 \\ X(0) = X(l) = 0 \end{cases} \quad X_n = \sin\frac{n\pi x}{l}$$

$$\sum_{n=1}^{\infty} T_n''(t)\sin\frac{n\pi x}{l} = a^2\sum_{n=1}^{\infty} T_n(t)\left(-\left(\frac{n\pi}{l}\right)^2\right)\sin\frac{n\pi x}{l} + f(x,t)$$

$$\sum_{n=1}^{\infty} \underbrace{\left(T_n'' + \left(\frac{an\pi}{l}\right)^2 T_n\right)}_{f_n(t)} \sin\frac{n\pi x}{l} = \underbrace{f(x,t)}$$

$$f_n(t) = \frac{1}{\|X_n\|^2}\int_0^l f(x,t)\,X_n(x)\,dx$$

$$T_n'' + \Big(\underbrace{\frac{an\pi}{l}}_{\omega_n}\Big)^2 T_n = \underbrace{\frac{2}{l}\int_0^l f(x,t)\sin\frac{n\pi x}{l}\,dx}_{= f_n(t)}, \quad t>0$$

$(4)\to(2)$:
$$u(x,0) \overset{(4)}{=} \sum_{n=1}^{\infty} T_n(0)\sin\frac{n\pi x}{l} \overset{(2)}{=} 0 \quad \Rightarrow \quad T_n(0)=0 \quad \forall n$$

$$u_t(x,0) \overset{(4)}{=} \sum_{n=1}^{\infty} T_n'(0)\sin\frac{n\pi x}{l} \overset{(2)}{=} 0 \quad \Rightarrow \quad T_n'(0)=0 \quad \forall n$$

$$(5) \quad \begin{cases} T_n'' + \omega_n^2 T_n = f_n(t), \quad t>0 \\ T_n(0) = T_n'(0) = 0 \end{cases}$$

$n = 1, 2, \ldots$

$$T_n'' + (\omega_n)^2 T_n = f_n(t)$$

1) $\lambda^2 + (\omega_n)^2 = 0, \quad \lambda = \pm i\omega_n, \quad \cos\omega_n t, \sin\omega_n t$

$$T_n^{одн}(t) = A_n \cos\omega_n t + B_n \sin\omega_n t$$

2) $T_n(t) = A_n(t)\cos\omega_n t + B_n(t)\sin\omega_n t \quad (*)$

$$?\ \begin{cases} A_n'(t)\cos\omega_n t + B_n'(t)\sin\omega_n t = 0 \\ -A_n'(t)\omega_n \sin\omega_n t + \omega_n B_n'(t)\cos\omega_n t = f_n(t) \end{cases}$$

$$(6)\quad T_n(t) = \frac{1}{\omega_n}\int_0^t f_n(\tau)\sin\omega_n(t-\tau)\,d\tau \quad , n = 1, 2 \ldots$$

$(6) \to (4)$

$$(1)\quad u_{tt} = a^2 u_{xx} + f(x,t), \quad 0 < x < l, \ t > 0$$

$$(2)\quad \begin{array}{l} u(x,0) = \varphi(x), \\ u_t(x,0) = \psi(x), \end{array} \quad 0 \le x \le l$$

$$(3)\quad \begin{array}{l} u(0,t) = \mu(t) \\ u(l,t) = \nu(t) \end{array} \quad t \geq 0$$

1. $(4)\quad u(x,t) = w(x,t) + v(x,t)$, $\quad w(x,t)$: $\quad w(0,t) = \mu(t), \ w(l,t) = \nu(t) \quad (5)$

$$w(x,t) = A(t)x + B(t) \quad (*)$$

$$w(0,t) \overset{(*)}{=} B(t) = \mu(t)$$

$$w(l,t) \overset{(*)}{=} A(t)\cdot l + \mu(t) = \nu(t)$$

$$A(t) = \frac{\nu(t) - \mu(t)}{l}$$

$$(6)\quad \boxed{w(x,t) = \frac{\nu(t) - \mu(t)}{l}\cdot x + \mu(t)}$$

2.

$$(4) \to (1),(2)$$
$$(3)$$

$$w_{tt} + v_{tt} = a^2 (w_{xx} + v_{xx}) + f(x,t)$$

$$(1') \qquad v_{tt} = a^2 v_{xx} + \underbrace{f(x,t) + a^2 w_{xx} - w_{tt}}_{= f_1(x,t)}$$

$$w(x,0) + v(x,0) = \varphi(x) \longrightarrow v(x,0) = \underbrace{\varphi(x) - w(x,0)}_{=: \varphi_1(x)}$$

$$w_t(x,0) + v_t(x,0) = \psi(x) \longrightarrow v_t(x,0) = \underbrace{\psi(x) - w_t(x,0)}_{=: \psi_1(x)}$$

$$\underbrace{w(0,t)}_{\overset{(6)}{=} \mu(t)} + v(0,t) = \mu(t) \longrightarrow v(0,t) = 0$$

$$\underbrace{w(\ell,t)}_{\overset{(6)}{=} \nu(t)} + v(\ell,t) = \nu(t) \longrightarrow v(\ell,t) = 0$$

3.

$$(7)\begin{cases} v_{tt} = a^2 v_{xx} + f_1(x,t), & 0<x<l, \quad t>0 \\ v(x,0) = \varphi_1(x) & \\ v_t(x,0) = \psi_1(x) & 0 \le x \le l \\ v(0,t) = v(l,t) = 0, & t \geq 0 \end{cases}$$

$$(8) \quad v(x,t) = \sum_n T_n(t) X_n(x) = \sum_{n=1}^{\infty} T_n(t) \sin\frac{n\pi x}{l} \to (7_1)$$

$$\omega_n = \frac{a\pi n}{l} \qquad \sum_{n=1}^{\infty} \left(T_n'' + \omega_n^2 T_n\right) \sin\frac{n\pi x}{l} = f_1(x,t)$$

$$(9)\; T_n'' + \omega_n^2 T_n = \frac{2}{l}\int_0^l f_1(x,t) \sin\frac{n\pi x}{l}\,dx, \quad t>0$$

$$(10)$$

$$(8) \to (7_2): \quad v(x,0) = \sum_{n=1}^{\infty} T_n(0) \sin\frac{n\pi x}{l} = \varphi_1(x) \qquad T_n(0) = \frac{2}{l}\int_0^l \varphi_1(x)\sin\frac{n\pi x}{l}\,dx = \varphi_{1n}$$

$$v_t(x,0) = \sum_{n=1}^{\infty} T_n'(0) \sin\frac{n\pi x}{l} = \psi_1(x), \qquad T_n'(0) = \frac{2}{l}\int_0^l \psi_1(x)\sin\frac{n\pi x}{l}\,dx = \psi_{1n}$$