Волновое уравнение на полупрямой. Метод продолжения

(1) $u_{tt} = a^2 u_{xx}$, $x > 0$, $t > 0$

(2) $u(x,0) = \varphi(x)$ $x \geqslant 0$
$u_t(x,0) = \psi(x)$

(3) $\alpha u_x(0,t) - \beta u(0,t) = 0$, $t \geqslant 0$

$\alpha, \beta \geqslant 0$, $\alpha + \beta \neq 0$

$\alpha \varphi'(0) - \beta \varphi(0) = 0$

$\alpha \psi'(0) - \beta \psi(0) = 0$

Задача Коши

$$\begin{cases} u_{tt} = a^2 u_{xx}, & x \in R^1,\ t > 0 \\ u(x,0) = \varphi(x) & \\ u_t(x,0) = \psi(x) & x \in R^1 \end{cases}$$

1) $\varphi(x), \psi(x)$ - нечет, $\qquad u(0,t) = 0$

2) $\varphi(x), \psi(x)$ - чет. $\qquad u_x(0,t) = 0$

3) $\left.\begin{array}{l} \alpha\varphi'(x) - \beta\varphi(x) \\ \alpha\psi'(x) - \beta\psi(x) \end{array}\right|$ нечет. $\qquad \alpha u_x(0,t) - \beta u(0,t) = 0$

$$
\begin{cases}
(1)\quad u_{tt} = a^2 u_{xx}, & x>0,\ t>0 \\
(2)\quad u(x,0) = \varphi(x) \\
\quad u_t(x,0) = \psi(x) & x \geqslant 0 \\
(3)\quad u(0,t) = 0, & t \geqslant 0
\end{cases}
$$

$$\underline{\varphi(0) = \psi(0) = 0}$$

$$(4)\quad \Phi(x) = \begin{cases} \varphi(x), & x \geqslant 0 \\ -\varphi(-x), & x<0 \end{cases} \qquad \Psi(x) = \begin{cases} \psi(x), & x \geqslant 0 \\ -\psi(-x), & x<0 \end{cases}$$

$$\boxed{x \geqslant 0,\ t \geqslant 0}$$

$U(x,t)$:

$$(5)\begin{cases} U_{tt} = a^2 U_{xx}, & x \in \mathbb{R}^1,\ t>0 \\ U(x,0) = \Phi(x) \\ U_t(x,0) = \Psi(x) & x \in \mathbb{R}^1 \end{cases}$$

$$\boxed{U(x,t) = \frac{\Phi(x+at)+\Phi(x-at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \Psi(\xi)\,d\xi} \quad (6)$$

$$\boxed{U(x,t) \overset{x \geqslant 0}{=} u(x,t)}$$

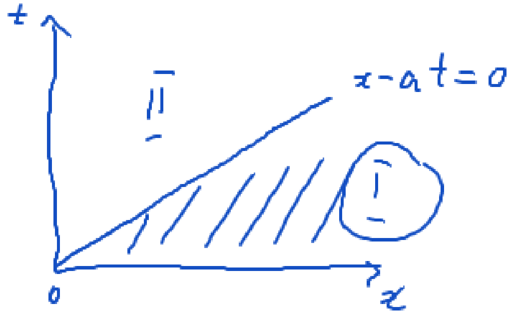
$$U(x,0) = \Phi(x) \overset{x \geqslant 0}{=} \varphi(x)$$

$$U_t(x,0) = \Psi(x) \overset{x \geqslant 0}{=} \psi(x)$$

$$\underset{\substack{\parallel \\ u(0,t)}}{U(0,t)} = 0\ ?$$

1) $x - at \geq 0$

$$u(x,t) = \frac{\varphi(x+at) + \varphi(x-at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\xi)\,d\xi \qquad (\mathrm{I})$$



2) $x - at < 0$

$$u(x,t) = \frac{\varphi(x+at) - \varphi(at-x)}{2} + \frac{1}{2a}\int_{at-x}^{x+at} \psi(\xi)\,d\xi \qquad \mathrm{II}$$

$$-\int_{x-at}^{0} \psi(-\xi)\,d\xi \qquad \int_{0}^{x+at} \psi(\xi)\,d\xi$$

$-\xi = \eta$

$$= -\int_{at-x}^{0} \psi(\eta)(-d\eta)$$

$$\int_{at-x}^{0} \psi(\eta)\,d\eta + \int_{0}^{x+at} \psi(\eta)\,d\eta$$

(1) $u_{tt} = a^2 u_{xx}, \quad x > 0, \ t > 0$

(2) $u(x,0) = \varphi(x)$, $u_t(x,0) = \psi(x)$ $\quad x \geq 0$ $\qquad \varphi'(0) = \psi'(0) = 0$

(3') $u_x(0,t) = 0, \quad t \geq 0$

$$\Phi(x) = \begin{cases} \varphi(x), & x \geq 0 \\ \varphi(-x), & x < 0 \end{cases} \qquad \Psi(x) = \begin{cases} \psi(x), & x \geq 0 \\ \psi(-x), & x < 0 \end{cases}$$

$$U_x(0,t) = 0$$

$\parallel$

$u_x(0,t)$

1) $x - at \geq 0$

$$u(x,t) = \frac{\varphi(x+at) + \varphi(x-at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\xi)\,d\xi$$

2) $x - at < 0$

$$u(x,t) = \frac{\varphi(x+at) + \varphi(at-x)}{2} + \frac{1}{2a}\left(\int_{x-at}^{0} \psi(-\xi)\,d\xi + \int_{0}^{x+at} \psi(\xi)\,d\xi\right)$$

$$-\int_{at-x}^{0} \psi(\xi)\,d\xi = \int_{0}^{at-x} \psi(\xi)\,d\xi$$

$$
\begin{aligned}
&(1) && \begin{cases} u_{tt} = a^2 u_{xx}, & x > 0,\ t > 0 \\ u(x,0) = \varphi(x) & x \geqslant 0 \\ u_t(x,0) = \psi(x) & \\ u_x(0,t) - h\,u(0,t) = 0, & t \geqslant 0 \end{cases} \\
&(2) && \\
&(3'') &&
\end{aligned}
$$

$$\underline{\varphi'(0) - h\varphi(0) = \psi'(0) - h\psi(0) = 0}$$

$$\Phi(x),\ \Psi(x):\quad \left.\begin{matrix} \Phi'(x) - h\Phi(x) \\ \Psi'(x) - h\Psi(x) \end{matrix}\right| \ \underline{\text{нечет.}}$$

$$\underline{\Phi(x) = \varphi(x)},\ x \geqslant 0, \qquad \Psi(x) = \psi(x),\ x \geqslant 0$$

$$\Phi(x),\ x < 0 \qquad \Phi'(x) - h\Phi(x) = -\left(\Phi'(-x) - h\Phi(-x)\right) \quad \forall\, x$$

$$(*) \begin{cases} \Phi'(x) - h\Phi(x) = \underbrace{-\left(\varphi'(-x) - h\varphi(-x)\right)}_{=\, f(x)}, & x < 0 \\ \Phi(0) = \varphi(0) \end{cases}$$

1) $\Phi'(x) - h\Phi(x) = 0 \qquad \lambda - h = 0 \qquad \lambda = h \qquad e^{\lambda x}$

$\Phi(x) = C \cdot e^{hx}$

2) $\underline{\Phi(x) = C(x)e^{hx}} \qquad C'(x)e^{hx} + C(x)he^{hx} - h\cdot C(x)e^{hx} = f(x)$

$C'(x) = f(x)e^{-hx} \qquad C(x) = \int_0^x f(\xi)e^{-h\xi}d\xi + C_0$

$$\Phi(x) = \int_0^x f(z)e^{h(x-z)}dz + C_0 e^{hx} \quad \Bigg| \quad C_0 = \varphi(0)$$

$$\Phi(0) = \varphi(0)$$

$$\Phi(x) = \int_0^x f(z)e^{h(x-z)}dz + \varphi(0)e^{hx} = -\int_0^x \left(\varphi'(-z) - h\varphi(z)\right) e^{h(x-z)}dz + \varphi(0)e^{hx} =$$

$$= -\boxed{\int_0^x \varphi'(-z)e^{h(x-z)}dz} + h\int_0^x \varphi(z)e^{h(x-z)}dz + \underline{\underline{\varphi(0)e^{hx}}} =$$

$$\int_0^x \varphi'(-z)e^{h(x-z)}dz = -\varphi(-z)e^{h(x-z)}\Big|_0^x + (-h)\int_0^x \varphi(-z)e^{h(x-z)}dz = \underline{-\varphi(-x) + \varphi(0)e^{hx} - h\int_0^x \varphi(-z)e^{h(x-z)}dz}$$

$$\varphi'(-z)dz = -d(\varphi(-z))$$

$$= \varphi(-x) + 2h\int_0^x \varphi(z)e^{h(x-z)}dz$$

$$\Phi(x) = \begin{cases} \varphi(x), & x \geq 0 \\ \varphi(-x) + 2h \int_0^x \varphi(-\xi) e^{h(x-\xi)} d\xi, & x < 0 \end{cases}$$

$$\Psi(x) = \begin{cases} \psi(x), & x \geq 0 \\ \psi(-x) + 2h \int_0^x \psi(-\xi) e^{h(x-\xi)} d\xi, & x < 0 \end{cases}$$