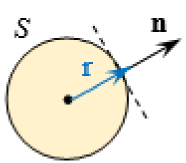
Условие разрешимости задачи Неймана для уравнения Лапласа. Примеры

Задача Неймана для уравнения Лапласа в круге:

$$\begin{cases} \Delta u = 0, \quad 0 \le r < a, \quad 0 \le \varphi < 2\pi, \\ \left.\dfrac{\partial u}{\partial r}\right|_{r=a} = f(\varphi) \end{cases}$$

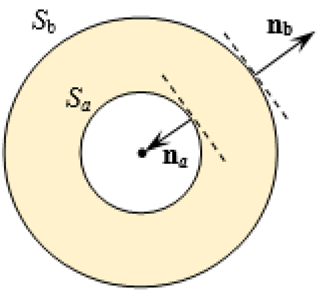


$$\int_S \frac{\partial u}{\partial n} ds = \int_0^{2\pi} \left.\frac{\partial u}{\partial r}\right|_{r=a} a d\varphi = a \int_0^{2\pi} f(\varphi) d\varphi$$

$$\int_S \frac{\partial u}{\partial n} ds = 0 \quad \Leftrightarrow \quad \boxed{\int_0^{2\pi} f(\varphi) d\varphi = 0}$$

Задача Неймана для уравнения Лапласа в кольце:

$$\begin{cases} \Delta u = 0, \quad a < r < b, \quad 0 \le \varphi < 2\pi, \\ \left.\dfrac{\partial u}{\partial r}\right|_{r=a} = f(\varphi), \quad \left.\dfrac{\partial u}{\partial r}\right|_{r=b} = g(\varphi) \end{cases}$$



$$\int_S \frac{\partial u}{\partial n} ds = \int_{S_a \cup S_b} \frac{\partial u}{\partial n} ds = \int_{S_a} \frac{\partial u}{\partial n_a} ds + \int_{S_b} \frac{\partial u}{\partial n_b} ds =$$

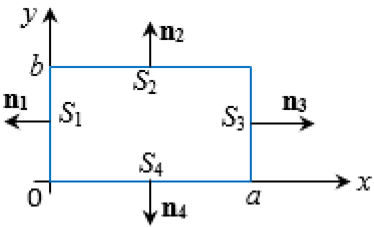
$$= \int_0^{2\pi} \left.\left(-\frac{\partial u}{\partial r}\right)\right|_{r=a} a d\varphi + \int_0^{2\pi} \left.\frac{\partial u}{\partial r}\right|_{r=b} b d\varphi =$$

$$= -a \int_0^{2\pi} f(\varphi) d\varphi + b \int_0^{2\pi} g(\varphi) d\varphi$$

$$\int_S \frac{\partial u}{\partial n} ds = 0 \quad \Leftrightarrow \quad \boxed{a\int_0^{2\pi} f(\varphi)d\varphi = b\int_0^{2\pi} g(\varphi)d\varphi}$$

Задача Неймана для уравнения Лапласа в прямоугольнике

$$\begin{cases} \Delta u = 0, \quad 0 < x < a, \quad 0 < y < b, \\ u_x(0,y) = f_1(y), \quad u_x(a,y) = f_2(y), \quad 0 \le y \le b, \\ u_y(x,0) = g_1(x), \quad u_y(x,b) = g_2(x), \quad 0 \le x \le a. \end{cases}$$



$$\int_S \frac{\partial u}{\partial n} ds = \int_{S_1\cup S_2\cup S_3\cup S_4} \frac{\partial u}{\partial n} ds = \int_{S_1} \frac{\partial u}{\partial n_1} ds + \int_{S_2} \frac{\partial u}{\partial n_2} ds + \int_{S_3} \frac{\partial u}{\partial n_3} ds + \int_{S_4} \frac{\partial u}{\partial n_4} ds =$$

$$= \int_0^b \left(-\frac{\partial u}{\partial x}\right)\bigg|_{x=0} dy + \int_0^a \frac{\partial u}{\partial y}\bigg|_{y=b} dx + \int_0^b \frac{\partial u}{\partial x}\bigg|_{x=a} dy + \int_0^a \left(-\frac{\partial u}{\partial y}\right)\bigg|_{y=0} dx =$$

$$= -\int_0^b f_1(y)dy + \int_0^a g_2(x)dx + \int_0^b f_2(y)dy - \int_0^a g_1(x)dx$$

$$\boxed{\int_S \frac{\partial u}{\partial n} ds = 0 \quad \Leftrightarrow \quad \int_0^b \big(f_1(y) - f_2(y)\big)dy + \int_0^a \big(g_1(x) - g_2(x)\big)dx = 0.}$$