Уравнения в полных дифференциалах.

Интегрирующий множитель

$$(1) \quad P(x,y)\,dx + Q(x,y)\,dy = 0 \qquad \frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$$

$$\mu = \mu(x,y)$$

$$(2) \quad \underbrace{\mu \cdot P}\,dx + \underbrace{\mu Q}\,dy = 0 \qquad \text{УПД}$$

$$\frac{\partial(\mu P)}{\partial y} \equiv \frac{\partial(\mu Q)}{\partial x}$$

$$\ln|\mu| = \int f(x)\,dx + \ln|c|$$

$$P \cdot \frac{\partial \mu}{\partial y} + \mu \frac{\partial P}{\partial y} = Q \frac{\partial \mu}{\partial x} + \mu \frac{\partial Q}{\partial x}$$

$$P \frac{\partial \mu}{\partial y} - Q \frac{\partial \mu}{\partial x} = \mu \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \qquad (3)$$

1) $\mu = \mu(x)$:

$$-Q\mu'(x) = \mu(x) \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$$

$$\mu'(x) = \mu(x) \frac{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}{-Q} = f(x)$$

$$\mu'(x) = \mu(x) f(x)$$

$$\frac{d\mu}{\mu} = f(x)\,dx \qquad c \neq 0$$

$$\mu(x) = C e^{\int f(x)\,dx} \qquad \exists\ \underline{C = 1}$$

$$\frac{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}{-Q} = \frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q} = f(x) \longrightarrow \mu = \mu(x): \quad \mu(x) = C \cdot e^{\int f(x)\,dx}$$

$(C - \forall \neq 0)$

2) $\mu = \mu(y)$

$$P\mu'(y) = \mu(y)\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$$\mu'(y) = \mu(y)\,\frac{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}{P} = f(y)$$

$$\mu'(y) = \mu(y) f(y)$$

$$\mu(y) = C \cdot e^{\int f(y)\,dy}$$

$C - \forall \neq 0$

$$\frac{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}{P} = f(y) \longrightarrow \mu = C \cdot e^{\int f(y)\,dy}$$

$C - \forall \neq 0$

№ 354 (Матвеев)

$$\underbrace{\left(\frac{x}{y}+1\right)}_{P(x,y)}dx+\underbrace{\left(\frac{x}{y}-1\right)}_{Q(x,y)}dy=0\ ,\quad y\neq 0$$

$$\frac{\partial P}{\partial y}=-\frac{x}{y^2}\neq\frac{\partial Q}{\partial x}=\frac{1}{y}$$

1) $\mu \overset{?}{=} \mu(x)$ $\quad \dfrac{-\frac{x}{y^2}-\frac{1}{y}}{\frac{x}{y}-1}\neq f(x)$

2) $\mu \overset{?}{=} \mu(y)$ $\quad \dfrac{\frac{1}{y}+\frac{x}{y^2}}{\frac{x}{y}+1}=\dfrac{\frac{1}{y}\left(1+\frac{x}{y}\right)}{\frac{x}{y}+1}=\frac{1}{y}=f(y)$

$$\mu(y)=C\cdot e^{\int\frac{1}{y}dy}=C\cdot e^{\ln|y|}=C\cdot|y|$$

$$\mu(y)=y$$

$$u=\frac{x^2}{2}+yx-\frac{y^2}{2}$$

√ $\underbrace{(x+y)}_{P'_y=1}dx+\underbrace{(x-y)}_{Q'_x=1}dy=0$ $\quad \equiv$

УПД

$$\begin{cases}\dfrac{\partial u}{\partial x}=x+y \xrightarrow{\int x} u=\dfrac{x^2}{2}+yx+\varphi(y)\\[2mm] \dfrac{\partial u}{\partial y}=x-y\end{cases}$$

$$x+\varphi'(y)=x-y \qquad \varphi'(y)=-y$$

$$\varphi(y)=-\frac{y^2}{2}+C_0$$

$C_0=0$

$$\boxed{\frac{x^2}{2}+yx-\frac{y^2}{2}=C}$$

$N\ 355\ (\mu)$

$$\underbrace{(x^2+y)}_{P(x,y)}dx \underbrace{-x}_{Q=-x}dy = 0$$

$x = 0$

$x^2 + y - x\frac{dy}{dx} = 0$ ✓

1) $\frac{\partial P}{\partial y} = 1 \neq \frac{\partial Q}{\partial x} = -1$

2) $\mu \overset{?}{=} \mu(x)$ $\qquad \dfrac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q} = \frac{2}{-x} = f(x)$

$$\mu(x) = C\cdot e^{\int\frac{2}{-x}dx} = C\cdot e^{-2\ln|x|} = C\cdot e^{-\ln x^2} = \frac{C}{x^2}$$

$\exists\ C = 1$

$\mu(x) = \frac{1}{x^2}$

3) $x = 0$ - реш.

4) $\frac{x^2+y}{x^2}dx - \frac{x}{x^2}dy = 0$

$\left(1 + \frac{y}{x^2}\right)dx - \frac{1}{x}dy = 0$

$$\begin{cases} \frac{\partial u}{\partial x} = 1 + \frac{y}{x^2} \\ \frac{\partial u}{\partial y} = -\frac{1}{x} \end{cases}$$

$\to u = x - \frac{y}{x} + \varphi(y)$

$-\frac{1}{x} + \varphi'(y) = -\frac{1}{x}$

$\varphi'(x) = 0 \to \varphi(x) = C_0$

$u = x - \frac{y}{x} + C_0$

$C_0 = 0$

$\boxed{x - \frac{y}{x} = C}$

Ответ:

$x = 0$

$x - \frac{y}{x} = C$

N 358 (Матвеев)

$$\underbrace{(x\cos y - y\sin y)}_{Q}dy + \underbrace{(x\sin y + y\cos y)}_{P}dx = 0$$

$$\frac{\partial P}{\partial y} = x\cos y + \cos y - y\sin y$$

$$\neq$$

$$\frac{\partial Q}{\partial x} = \cos y$$

μ – ? $\mu(x)$?

$$\frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q} = \frac{x\cos y - y\sin y}{x\cos y - y\sin y} = 1 \text{ “} f(x) \text{”}$$

$$\mu(x) = C\cdot e^{\int 1\,dx} = C\cdot e^{x}, \quad \exists\, C = 1 \qquad \mu(x) = e^{x}$$

$$e^{x}(x\cos y - y\sin y)dy + e^{x}(x\sin y + y\cos y)dx = 0 \qquad \text{УПД}$$

$$\begin{cases} \dfrac{\partial u}{\partial x} = e^{x}(x\sin y + y\cos y) \xrightarrow{\int, x} u = \sin y \underbrace{\int e^{x}x\,dx}_{e^{x}(x-1)} + y\cos y \underbrace{\int e^{x}dx}_{e^{x}} + \varphi(y) = e^{x}(x\sin y + y\cos y - \sin y) + \varphi(y) \\[2ex] \dfrac{\partial u}{\partial y} = e^{x}(x\cos y - y\sin y) \end{cases}$$

$$e^{x}(x\cos y - y\sin y + \underline{\cos y} - \underline{\cos y}) + \varphi'(y) = e^{x}(x\cos y - y\sin y) \longrightarrow \varphi'(y) = 0, \quad \varphi(y) = C_0$$

$C_0 = 0$

$$\boxed{e^{x}(x\sin y + y\cos y - \sin y) = C}$$

№356 [M]

$$(2xy^2 - y)\,dx + (y^2 + x + y)\,dy = 0$$